The Weihrauch lattice around ATR_0 and $\mathbf{\Pi}^1_1$-CA_0

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The project

In a 2015 Dagstuhl seminar I asked "What do the Weihrauch hierarchies look like once we go to very high levels of reverse mathematics strength?"

In other words, I proposed to study the multi-valued functions arising from theorems which lie around ATR_0 and $\mathbf{\Pi}^1_1$-CA_0.

People who have contributed to this project include Takayuki Kihara, Arno Pauly, Jun Le Goh, Jeff Hirst, Paul-Elliot Anglès d'Auriac, and my students Manlio Valenti and Vittorio Cipriani.

Outline

1. **Weihrauch reducibility**
2. **Earlier results around** ATR_0
3. **The clopen and open Ramsey theorem**
4. **Arithmetic Weihrauch reducibility**
5. **Recent results around** $\mathbf{\Pi}^1_1$-CA_0

Represented spaces

A representation σ_X of a set X is a surjective partial function $\sigma_X : \subseteq \mathbb{N}^{\mathbb{N}} \to X$.

The pair (X, σ_X) is a represented space.

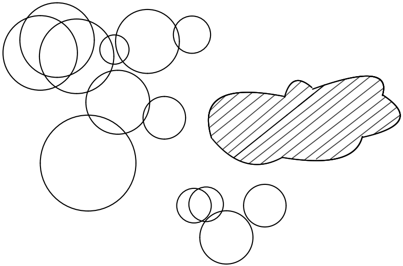
If $x \in X$ a σ_X-name for x is any $p \in \mathbb{N}^{\mathbb{N}}$ such that $\sigma_X(p) = x$.

Representations are analogous to the codings used in reverse mathematics to speak about various mathematical objects in subsystems of second order arithmetic.

The negative representation of closed sets

Let (X, α, d) be a computable metric space.

In the negative representation of the set $\mathcal{A}^-(X)$ of closed subsets of X a name for the closed set C is a sequence of open balls with center in D and rational radius whose union is $X \setminus C$.



When $X = \mathbb{N}^\mathbb{N}$ or $X = 2^\mathbb{N}$ the negative representation is computably equivalent to the representation of C by a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $[T] = C$.

Realizers

If (X, σ_X) and (Y, σ_Y) are represented spaces and $f :\subseteq X \rightrightarrows Y$ a realizer for f is a function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $\sigma_Y(F(p)) \in f(\sigma_X(p))$ whenever $f(\sigma_X(p))$ is defined, i.e. whenever p is a name of some $x \in \mathrm{dom}(f)$.

We write $F \vdash f$.

$$\begin{tikzcd}
p \in \mathbb{N}^\mathbb{N} \arrow[r, "F"] \arrow[d, "\sigma_X"'] & F(p) \in \mathbb{N}^\mathbb{N} \arrow[d, "\sigma_Y"] \\
x \in X \arrow[r, Rightarrow, "f"'] & y \in f(x)
\end{tikzcd}$$

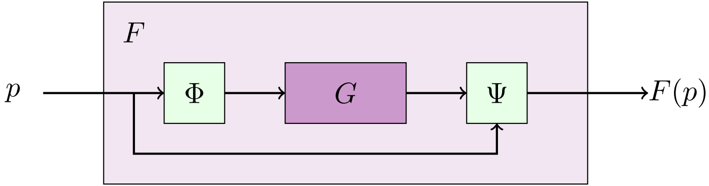
Notice that different names of the same $x \in \mathrm{dom}(f)$ might be mapped by F to names of different elements of $f(x)$.

f is computable if it has a computable realizer.

Weihrauch reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions between represented spaces.

f is Weihrauch reducible to g if there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $G \vdash g$ we have that $F \vdash f$ where $F(p) = \Psi(p, G\Phi(p))$.



In other words, for all names $p \in \mathbb{N}^{\mathbb{N}}$ for some $x \in \mathrm{dom}(f)$, we have that $\Phi(p)$ is a name for some element of $\mathrm{dom}(g)$ and $\Psi(p, G\Phi(p))$ is a name for some element of $f(x)$.

We write $f \leq_{\mathrm{W}} g$.

The Weihrauch lattice

$\leq_{\mathrm{W}}$ is reflexive and transitive and induces the equivalence relation $\equiv_{\mathrm{W}}$. The $\equiv_{\mathrm{W}}$-equivalence classes are called Weihrauch degrees.

The partial order on the sets of Weihrauch degrees is a distributive bounded lattice with several natural and useful algebraic operations: the Weihrauch lattice.

Products

The parallel product of $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ is $f \times g :\subseteq X \times Z \rightrightarrows Y \times W$ defined by

$$(f \times g)(x, z) = f(x) \times g(z).$$

The compositional product $f \star g$ satisfies

$$f \star g \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{f_1 \circ g_1 \mid f_1 \leq_{\mathrm{W}} f \wedge g_1 \leq_{\mathrm{W}} g\}$$

and thus is the hardest problem that can be realized using first g, then something computable, and finally f.

Parallelization

If $f :\subseteq X \rightrightarrows Y$ is a multi-valued function, the (infinite) parallelization of f is the multi-valued function $\widehat{f} : X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$ with $\mathrm{dom}(\widehat{f}) = \mathrm{dom}(f)^{\mathbb{N}}$ defined by $f((x_n)_{n\in\mathbb{N}}) = \prod_{n\in\mathbb{N}} f(x_n)$.

$\widehat{f}$ computes f countably many times in parallel.

f is parallelizable if $\widehat{f} \equiv_{\mathrm{W}} f$.

The finite parallelization of f is the multi-valued function $f^* : X^* \rightrightarrows Y^*$ where $X^* = \bigcup_{i\in\mathbb{N}}(\{i\} \times X^i)$ with $\mathrm{dom}(f^*) = \mathrm{dom}(f)^*$ defined by $f^*(i, (x_j)_{j<i}) = \{i\} \times \prod_{j<i} f(x_j)$.

Some examples

- The limited principle of omniscience is the function $\mathsf{LPO} : \mathbb{N}^{\mathbb{N}} \to 2$ such that $\mathsf{LPO}(p) = 0$ iff $\forall i\, p(i) = 0$.
- $\lim : \subseteq (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ maps a convergent sequence in Baire space to its limit.

$\lim$ is parallelizable, while LPO is not (and in fact $\widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \lim$).

Choice functions

Let X be a computable metric space and recall that $\mathcal{A}^-(X)$ is the space of its closed subsets represented by negative information.

$\mathsf{C}_X :\subseteq \mathcal{A}^-(X) \rightrightarrows X$ is the choice function for X: it picks from a nonempty closed set in X one of its elements.

$\mathsf{UC}_X :\subseteq \mathcal{A}^-(X) \to X$ is the unique choice function for X: it picks from a singleton (represented as a closed set) in X its unique element (in other words, UC_X is the restriction of C_X to singletons).

$\mathsf{TC}_X : \mathcal{A}^-(X) \rightrightarrows X$ is the total continuation of the choice function for X: it extends C_X by setting $\mathsf{TC}_X(\emptyset) = X$.

In general we have $\mathsf{UC}_X \leq_{\mathrm{W}} \mathsf{C}_X \leq_{\mathrm{W}} \mathsf{TC}_X$ and, for example, $\mathsf{C}_{\mathbb{N}} <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}$ and $\mathsf{C}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{TC}_{2^{\mathbb{N}}}$.

Forms of choice on Baire space

It is important for us that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_\mathrm{W} \mathsf{C}_{\mathbb{N}^\mathbb{N}} <_\mathrm{W} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$.

The strictness of the first inequality follows from classical facts: there exists a computable tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with no hyperarithmetic paths, but if $[T]$ is countable (in particular if it is a singleton) then all paths are hyperarithmetic.

The strictness of the second inequality follows from the fact that to decide whether a tree is ill-founded we can first apply $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$ and then check the result using LPO.

The Weihrauch lattice and reverse mathematics

We can locate theorems in the Weihrauch lattice by looking at the multi-valued functions they naturally translate into.

In most cases the Weihrauch lattice refines the classification provided by reverse mathematics: statements which are equivalent over RCA_0 may give rise to functions with different Weihrauch degrees.

Weihrauch reducibility is finer because requires both uniformity and use of a single instance of the harder problem.

We have a good understanding of the connection between reverse mathematics and the Weihrauch lattice for levels up to ACA_0:

- computable functions correspond to RCA_0;
- $\mathsf{C}_{2^{\mathbb{N}}}$ corresponds to WKL_0;
- $\lim$ and its iterations correspond to ACA_0.

Arithmetical Transfinite Recursion

ATR is the function producing, for a well-order X, a jump hierarchy along X.

Theorem (Kihara-M-Pauly)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{ATR}$.

ATR_2 is the function producing, for a linear order X, either a jump hierarchy along X or a descending sequence in X.

Theorem (Goh)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{ATR}_2 <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Comprehension functions around ATR_0 and $\mathbf{\Pi}^1_1$-CA_0

Tr is the set of subtrees of $\mathbb{N}^{<\mathbb{N}}$.

If $T \in \mathrm{Tr}$ then $[T]$ is the set of the infinite paths through T.

- $\mathbf{\Sigma}^1_1$-Sep $: \subseteq (\mathrm{Tr} \times \mathrm{Tr})^{\mathbb{N}} \rightrightarrows 2^{\mathbb{N}}$ has domain $\{ (S_n, T_n)_{n\in\mathbb{N}} \mid \forall n \neg([S_n] \neq \emptyset \wedge [T_n] \neq \emptyset) \}$ and maps $(S_n, T_n)_{n\in\mathbb{N}}$ to $\{ f \in 2^{\mathbb{N}} \mid \forall n([S_n] \neq \emptyset \to f(n) = 0) \wedge ([T_n] \neq \emptyset \to f(n) = 1) \}$. ATR_0
- $\mathbf{\Delta}^1_1$-CA is the restriction of $\mathbf{\Sigma}^1_1$-Sep to $\{ (S_n, T_n)_{n\in\mathbb{N}} \mid \forall n([S_n] = \emptyset \leftrightarrow [T_n] \neq \emptyset) \}$. $< \mathsf{ATR}_0$
- $\chi_{\Pi^1_1} : \mathrm{Tr} \to 2$ such that $\chi_{\Pi^1_1}(T) = 0$ iff T is ill-founded.
- $\mathbf{\Pi}^1_1$-CA $= \widehat{\chi_{\Pi^1_1}}$ maps $(T_n)_{n\in\mathbb{N}}$ to the characteristic function of $\{ n \in \mathbb{N} \mid [T_n] \neq \emptyset \}$. $\mathbf{\Pi}^1_1$-CA_0

Theorem (Kihara-M-Pauly)

$\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathbf{\Sigma}^1_1$-Sep $\equiv_{\mathrm{W}} \mathbf{\Delta}^1_1$-CA.

Comparability of well-orders

WO is the set of well-orders on $\mathbb{N}$.

- $\mathsf{CWO} : \mathrm{WO} \times \mathrm{WO} \to \mathbb{N}^{\mathbb{N}}$ is the function that maps a pair of well-orders to the order preserving map from one of them onto an initial segment of the other. ATR_0
- $\mathsf{WCWO} : \mathrm{WO} \times \mathrm{WO} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is the multi-valued function that maps a pair of well-orders to the set of order preserving maps from one of them to the other. ATR_0

Theorem (Kihara-M-Pauly)

$\mathsf{CWO} \equiv_{\mathrm{W}} \widehat{\mathsf{WCWO}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$.

Theorem (Goh)

$\mathsf{WCWO} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$.

The perfect tree theorem

The Perfect Tree Theorem asserts that if $T \in \mathrm{Tr}$, then either $[T]$ is countable or T has a perfect subtree.

The conclusion of the theorem has the form $A \vee B$. To make it into a problem we have different options:

- $\mathsf{PTT}_1 :\subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr}$ is the multi-valued function that maps a tree with uncountably many paths to the set of its perfect subtrees. ATR_0
- $\mathsf{List} :\subseteq \mathrm{Tr} \rightrightarrows (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ is the multi-valued function that maps a tree with no perfect subtree to a list of its paths. ATR_0
- $\mathsf{PTT}_2 :\subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ is the multi-valued function that maps a tree to a pair $(T', (p_n))$ such that either T' is a perfect subtree of T or (p_n) lists all elements of $[T]$. ATR_0

Theorem (Kihara-M-Pauly)

$$\mathsf{List} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{PTT}_1 \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{PTT}_2 <_{\mathrm{W}}$$

$$<_{\mathrm{W}} \mathsf{TC}^*_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PTT}^*_2 <_{\mathrm{W}} \mathbf{\Pi}^1_1\text{-}\mathsf{CA}.$$

Open determinacy

We consider two-player perfect information games where two players alternate playing in $\mathbb{N}$. Open determinacy asserts that if the winning set for the first player is open, then one of the players has a winning strategy.

Again there are different ways of making it into a problem:

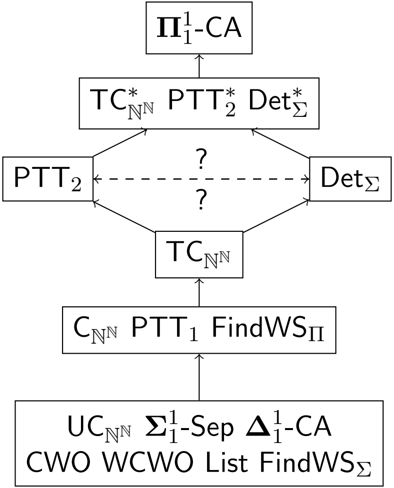
- $\mathsf{FindWS}_\Sigma :\subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr}$ is the multi-valued function that maps a tree T to the set of winning strategies for Player 1 in the game where Player 1 wins if the sequence constructed by the players $\notin [T]$. ATR_0
- $\mathsf{FindWS}_\Pi :\subseteq \mathrm{Tr} \rightrightarrows \mathrm{Tr}$ is the multi-valued function that maps a tree T to the set of winning strategies for Player 2 in the game above. ATR_0
- $\mathsf{Det}_\Sigma : \mathrm{Tr} \rightrightarrows \mathrm{Tr} \times \mathrm{Tr}$ is the multi-valued function that maps a tree T to a pair of strategies for the two Players such that one of them is winning in the game above. ATR_0

Results about open determinacy

Theorem (Kihara-M-Pauly)

$$\mathsf{FindWS}_\Sigma \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{FindWS}_\Pi \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}} <_W$$
$$<_W \mathsf{TC}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{Det}_\Sigma <_W \mathsf{TC}^*_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{Det}^*_\Sigma <_W \mathbf{\Pi}^1_1\text{-}\mathsf{CA}.$$

Recap



Goh studied also König's duality theorem in this context, and Kihara and Anglès D'Auriac have results about the functions corresponding to $\mathbf{\Sigma}^1_1$-AC_0 and $\mathbf{\Sigma}^1_1$-DC_0.

Spaces of infinite sets

We work in the space $[\mathbb{N}]^{\mathbb{N}}$ of infinite subsets of $\mathbb{N}$.
A member of $[\mathbb{N}]^{\mathbb{N}}$ can be identified with the strictly increasing function that enumerates it.

If $X \in [\mathbb{N}]^{\mathbb{N}}$ then $[X]^{\mathbb{N}}$ is the set of infinite subsets of X.
Notice that if f (increasingly) enumerates X, then $[X]^{\mathbb{N}} = \{\, f \cdot g \mid g \text{ is strictly increasing} \,\}$.

Every $[X]^{\mathbb{N}}$, and in particular $[\mathbb{N}]^{\mathbb{N}}$, is a closed subspace of $\mathbb{N}^{\mathbb{N}}$.
Thus $[X]^{\mathbb{N}}$ is a Polish space, and in fact is isometric to $\mathbb{N}^{\mathbb{N}}$.

Homogeneous sets

If $P \subseteq [\mathbb{N}]^{\mathbb{N}}$ we let

$$\begin{aligned} H(P) &= \{ X \in [\mathbb{N}]^{\mathbb{N}} \mid [X]^{\mathbb{N}} \subseteq P \vee [X]^{\mathbb{N}} \cap P = \emptyset \} \\ &= \{ f \in [\mathbb{N}]^{\mathbb{N}} \mid \forall g (f \cdot g \in P) \vee \forall g (f \cdot g \notin P) \}. \end{aligned}$$

The elements of $H(P)$ are called homogeneous sets for P.
If $[X]^{\mathbb{N}} \subseteq P$ then X lands in P.
If $[X]^{\mathbb{N}} \cap P = \emptyset$ then X avoids P.

Notice that a given P can have both homogeneous sets landing in P and homogeneous sets avoiding P.

P is Ramsey if $H(P) \neq \emptyset$, i.e. if there exist homogeneous sets for P.

Which subsets of $[\mathbb{N}]^{\mathbb{N}}$ are Ramsey?

- Every clopen set is Ramsey (Nash-Williams)
- Every Borel set is Ramsey (Galvin-Prikry)
- Every analytic set is Ramsey (Silver)
- (ZFC + measurable cardinals) Every $\mathbf{\Sigma}^1_2$ set is Ramsey (Silver)
- (ZF + $\mathsf{AD}_{\mathbb{R}}$) Every set is Ramsey (Prikry)

The reverse mathematics of the infinite Ramsey theorem

• Every clopen set is Ramsey	ATR_0
• Every open set is Ramsey	ATR_0
• Every $\mathbf{\Delta}^0_2$ set is Ramsey	$\mathbf{\Pi}^1_1$-CA_0
• Every Borel set is Ramsey	$\mathbf{\Pi}^1_1$-TR_0
• Every analytic set is Ramsey	$\mathbf{\Sigma}^1_1$-MI_0

Some observations about the open Ramsey theorem

Fix $P \subseteq [\mathbb{N}]^{\mathbb{N}}$ open.

- The set of elements of $H(P)$ which avoid P is closed; given a name $\langle P \rangle$ for P it is easy to define a tree $T_{\langle P \rangle}$ such that $[T_{\langle P \rangle}]$ is precisely this set.
- The set of elements of $H(P)$ which land in P is $\mathbf{\Pi}^1_1$; it can be $\mathbf{\Pi}^1_1$-complete.

The ATR_0 proof of open determinacy in Simpson's book proceeds by assuming that there is no set avoiding P and using the well-foundedness of $T_{\langle P \rangle}$ to construct a set landing in P.

This proof is asymmetric: to find a set avoiding P it suffices to find a path in $T_{\langle P \rangle}$ (even if there are sets landing in P), yet it gives no clue about building a set landing in P when there exist sets avoiding P.

Representing open and clopen sets

$\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})$ is the represented space of open subsets of $[\mathbb{N}]^\mathbb{N}$.

A name for $P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})$ is a list of finite strictly increasing sequences (σ_i) such that $X \in P$ if and only if $\exists i\, \sigma_i \sqsubset X$.

This representation is equivalent to representing $[\mathbb{N}]^\mathbb{N} \setminus P$ as an element of $\mathcal{A}^-([\mathbb{N}]^\mathbb{N})$.

$\mathbf{\Delta}^0_1([\mathbb{N}]^\mathbb{N})$ is the represented space of clopen subsets of $[\mathbb{N}]^\mathbb{N}$.

A name for $D \in \mathbf{\Delta}^0_1([\mathbb{N}]^\mathbb{N})$ consists of two names for members of $\mathbf{\Sigma}^0_1([\mathbb{N}]^\mathbb{N})$: one for D and one for $[\mathbb{N}]^\mathbb{N} \setminus D$.

This representation is equivalent to representing D and $[\mathbb{N}]^\mathbb{N} \setminus D$ as elements of $\mathcal{A}^-([\mathbb{N}]^\mathbb{N})$.

Multi-valued functions associated to the open Ramsey theorem

full $\mathbf{\Sigma}^0_1$-RT : $\mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ defined by $\mathbf{\Sigma}^0_1$-RT$(P) = H(P)$;

strong open $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1} :\subseteq \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ defined by
$\mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}) = \{\, P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \mid H(P) \cap P \neq \emptyset \,\}$ and
$\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}(P) = H(P) \cap P$;

strong closed $\mathsf{FindHS}_{\mathbf{\Pi}^0_1} :\subseteq \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ defined by
$\mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Pi}^0_1}) = \{\, P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \mid H(P) \nsubseteq P \,\}$ and
$\mathsf{FindHS}_{\mathbf{\Pi}^0_1}(P) = H(P) \setminus P$;

weak open $\mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}$ is the restriction of $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ to
$\{\, P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \mid H(P) \subseteq P \,\}$;

weak closed $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ is the restriction of $\mathsf{FindHS}_{\mathbf{\Pi}^0_1}$ to
$\{\, P \in \mathbf{\Sigma}^0_1([\mathbb{N}]^{\mathbb{N}}) \mid H(P) \cap P = \emptyset \,\}$.

Multi-valued functions associated to the clopen Ramsey theorem

full $\mathbf{\Delta}^0_1$-RT : $\mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ defined by $\mathbf{\Delta}^0_1$-RT$(D) = H(D)$;

strong $\mathsf{FindHS}_{\mathbf{\Delta}^0_1} :\subseteq \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \rightrightarrows [\mathbb{N}]^{\mathbb{N}}$ defined by $\mathrm{dom}(\mathsf{FindHS}_{\mathbf{\Delta}^0_1}) = \{D \in \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \mid H(D) \cap D \neq \emptyset\}$ and $\mathsf{FindHS}_{\mathbf{\Delta}^0_1}(D) = H(D) \cap D$;

weak $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$ is the restriction of $\mathsf{FindHS}_{\mathbf{\Delta}^0_1}$ to $\{\, D \in \mathbf{\Delta}^0_1([\mathbb{N}]^{\mathbb{N}}) \mid H(D) \subseteq D \,\}$.

Equivalences with $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$

Theorem (M-Valenti)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1} \equiv_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{W}} \mathbf{\Delta}^0_1\text{-}\mathsf{RT}$.

To show $\mathbf{\Delta}^0_1\text{-}\mathsf{RT} \le_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}$, given a name (D_0, D_1) for $D \in \mathbf{\Delta}^0_1([\mathbb{N}]^\mathbb{N})$ we uniformly compute a name for

$$E = \{ f \in [\mathbb{N}]^\mathbb{N} \mid \exists \sigma, \tau \in D_0 (\sigma^\frown\tau \sqsubset f) \} \cup \{ f \in [\mathbb{N}]^\mathbb{N} \mid \exists \sigma, \tau \in D_1 (\sigma^\frown\tau \sqsubset f) \} \in \mathbf{\Delta}^0_1([\mathbb{N}]^\mathbb{N}).$$

Then $H(E) = H(D)$ and $H(E) \subseteq E$ and hence $\mathsf{wFindHS}_{\mathbf{\Delta}^0_1}(E) = \mathbf{\Delta}^0_1\text{-}\mathsf{RT}(D)$.

$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$ is almost $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$

Theorem (M-Valenti)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{2^\mathbb{N}} \star \mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$.

Equivalences with $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$

Theorem (M-Valenti)

$\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$.

Σ^0_1-RT **is fairly strong**

Theorem (M-Valenti)

$\boldsymbol{\Sigma}^0_1$-RT $\not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$ *and* wFindHS$_{\boldsymbol{\Pi}^0_1} <_{\mathrm{W}} \boldsymbol{\Sigma}^0_1$-RT*.*

$\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is very strong

Theorem (M-Valenti)

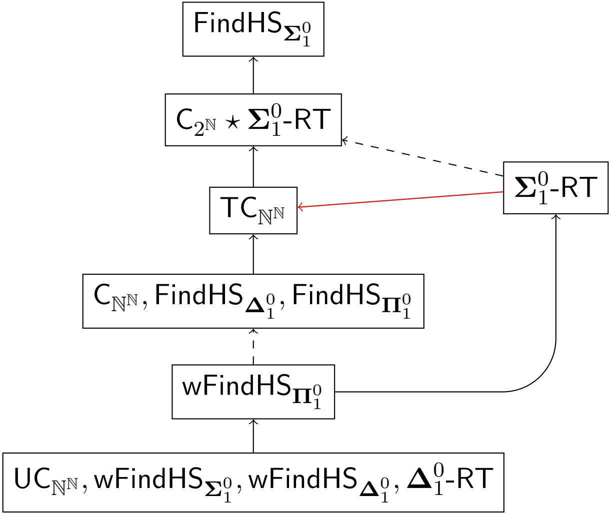
$\mathbf{\Sigma}^0_1$-$\mathsf{RT} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$, $\mathsf{TC}_{\mathbb{N}^\mathbb{N}} \times \mathsf{C}_{\mathbb{N}^\mathbb{N}} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$, $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \star \mathbf{\Sigma}^0_1$-$\mathsf{RT} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ *and* $\chi_{\Pi^1_1} <_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

Thus $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ escapes the levels of complexity found so far for multi-valued functions connected to ATR_0 and approaches $\mathbf{\Pi}^1_1$-CA_0. We do not know whether $\mathbf{\Pi}^1_1$-$\mathsf{CA} \leq_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

It is however true that the restatement of the open Ramsey theorem arising from $\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$ is quite unnatural:

> *if P is open and not all homogenous sets avoid P,*
> *then there exists an homogenous set landing in P.*

Recap


$\mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$
$\mathsf{C}_{2^{\mathbb{N}}} \star \mathbf{\Sigma}^0_1\text{-}\mathsf{RT}$
$\mathbf{\Sigma}^0_1\text{-}\mathsf{RT}$
$\mathsf{TC}_{\mathbb{N}^{\mathbb{N}}}$
$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}, \mathsf{FindHS}_{\mathbf{\Delta}^0_1}, \mathsf{FindHS}_{\mathbf{\Pi}^0_1}$
$\mathsf{wFindHS}_{\mathbf{\Pi}^0_1}$
$\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}, \mathsf{wFindHS}_{\mathbf{\Sigma}^0_1}, \mathsf{wFindHS}_{\mathbf{\Delta}^0_1}, \mathbf{\Delta}^0_1\text{-}\mathsf{RT}$

Arithmetic Weihrauch reducibility

Let $f :\subseteq X \rightrightarrows Y$, $g :\subseteq Z \rightrightarrows W$ be partial multi-valued functions between represented spaces. We say that f is arithmetically Weihrauch reducible to g, and we write $f \leq^a_{\mathrm{W}} g$ if there are arithmetic $\Phi, \Psi : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ such that for all $G \vdash g$ we have that $F \vdash f$ where $F(p) = \Psi(p, G\Phi(p))$.
Here a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is arithmetic if $F \leq_{\mathrm{W}} \lim^{(n)}$ for some $n \in \mathbb{N}$.

It is immediate that $f \leq_{\mathrm{W}} g$ implies $f \leq^a_{\mathrm{W}} g$.

Arithmetic Weihrauch reducibility was introduced by Kihara-Anglès D'Auriac and independently by Goh.

Perhaps this is the right reducibility for multi-valued functions above ACA_0.

Some arithmetic results

Theorem (M-Valenti)

- $\mathsf{wFindHS}_{\mathbf{\Pi}^0_1} \equiv^a_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$;
- $\mathsf{C}_{\mathbb{N}^\mathbb{N}} <^a_{\mathrm{W}} \mathbf{\Sigma}^0_1\text{-}\mathsf{RT} \equiv^a_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$;
- $\mathbf{\Sigma}^0_1\text{-}\mathsf{RT} <^a_{\mathrm{W}} \mathsf{FindHS}_{\mathbf{\Sigma}^0_1}$.

Recap under arithmetic Weihrauch reducibility

$$
\begin{tikzcd}
\boxed{\mathsf{FindHS}_{\boldsymbol{\Sigma}^0_1}} \\
\boxed{\boldsymbol{\Sigma}^0_1\text{-}\mathsf{RT}} \arrow[u] \\
\boxed{\mathsf{TC}_{\mathbb{N}^\mathbb{N}}} \arrow[u] \\
\boxed{\mathsf{C}_{\mathbb{N}^\mathbb{N}}, \mathsf{FindHS}_{\boldsymbol{\Delta}^0_1}, \mathsf{FindHS}_{\boldsymbol{\Pi}^0_1}, \mathsf{wFindHS}_{\boldsymbol{\Pi}^0_1}} \arrow[u] \\
\boxed{\mathsf{UC}_{\mathbb{N}^\mathbb{N}}, \mathsf{wFindHS}_{\boldsymbol{\Sigma}^0_1}, \mathsf{wFindHS}_{\boldsymbol{\Delta}^0_1}, \boldsymbol{\Delta}^0_1\text{-}\mathsf{RT}} \arrow[u]
\end{tikzcd}
$$

Perfect kernels

The study of multi-valued functions arising from theorems equivalent to $\mathbf{\Pi}^1_1$-CA_0 is in its infancy.

Let $\mathsf{PK}_{\mathrm{Tr}} : \mathrm{Tr} \to \mathrm{Tr}$ be the function that maps a tree T to its perfect kernel, i.e. the largest perfect subtree of T. $\mathbf{\Pi}^1_1$-CA_0

Theorem (Hirst)

$\mathbf{\Pi}^1_1$-$\mathsf{CA} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathrm{Tr}}$.

Let $\mathsf{PK}_{2^{\mathbb{N}}} : \mathcal{A}^-(2^{\mathbb{N}}) \to \mathcal{A}^-(2^{\mathbb{N}})$ and $\mathsf{PK}_{\mathbb{N}^{\mathbb{N}}} : \mathcal{A}^-(\mathbb{N}^{\mathbb{N}}) \to \mathcal{A}^-(\mathbb{N}^{\mathbb{N}})$ be the functions mapping a closed set C to its perfect kernel, i.e. the largest perfect closed subset of C. $\mathbf{\Pi}^1_1$-CA_0

Theorem (Cipriani-M-Valenti)

$\mathsf{PK}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathbf{\Pi}^1_1$-$\mathsf{CA}$ *and* $\mathbf{\Pi}^1_1$-$\mathsf{CA} \leq_{\mathrm{W}} \lim \star \mathsf{PK}_{\mathbb{N}^{\mathbb{N}}}$. *Thus* $\mathsf{PK}_{\mathbb{N}^{\mathbb{N}}} \not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ *and* $\mathbf{\Pi}^1_1$-$\mathsf{CA} \equiv^a_{\mathrm{W}} \mathsf{PK}_{\mathrm{Tr}} \equiv^a_{\mathrm{W}} \mathsf{PK}_{2^{\mathbb{N}}} \equiv^a_{\mathrm{W}} \mathsf{PK}_{\mathbb{N}^{\mathbb{N}}}$.

We do not know whether $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{PK}_{\mathbb{N}^{\mathbb{N}}}$.

The end

Thank you for your attention!